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Classical and Modified Mathematical Models of Electric Arc

Abstract: The article presents classical mathematical models of electric arc characterised by undetermined or unreduced ignition voltage. The aforementioned models include the well-known Mayr and Cassie models and their well-known Schwartz extensions with the power functions of the Mayr and Cassie voltage as well as the increased dissipation of energy within the high-current range. The modified models contain residual conductance, enabling the determination of arc ignition voltage. A similar approach was applied to modify the approximating functions of voltage and current characteristics used in the Pentegov model. Computer simulations were used to examine the influence of the parameters of the mathematical models on the shapes of static and dynamic characteristics of arc located in a circuit with sinusoidal current excitation.

Keywords: electric arc, mathematical models of electric arc, Mayr model, Cassie model

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Introduction

Simple mathematical models of arc are characterised by the low order of a differential equation (usually first order), linearity and a low number of parameters. Sometimes, activities undertaken to improve approximating models lead to the formation of the non-linearity of an equation through the variation of previously constant parameters or taking into consideration the additional dissipation of energy within the high-current range. However, the above-named approach usually maintains the indeterminacy of arc ignition voltage. In some simulations [1, 2], such indeterminacy of the model impedes the precise representation of high-frequency harmonics present in circuits with arc powered by alternating current. For this reason, the simplification of differential formulas is applied (e.g. the Schellhase model) [3, 4] at the expense of the negligence of energy balance conditions in the plasma chamber. Publications [5-8] present the tests of the Pentegov model, where appropriate static characteristics were applied. The use of the above-named characteristics ensured the determinacy of arc ignition voltage. However, the aforesaid approach somehow impeded the physical interpretation of processes simulated in circuits with arc. This article presents tests of classical and modified mathematical models provided with an additional component identifying the value of current abscissa, corresponding to ignition voltage.

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Modifications of classical dynamic models of electric arc

The adoption of various preliminary assumptions concerning the method of heat dissipation from the plasma column leads to the development of various mathematical models of arc. The extent, to which such assumptions can be fulfilled depends on excitation (range of current amplitude and frequency) and other arc burning physical conditions (e.g. temperature, pressure, the chemical composition of gas) [9-10]. The objective of the implementation of modifications in classical models or combining them to form hybrid models [11] is the extension of their application ranges when simulating electromagnetic processes in technological devices. Table 1 presents a set of differential equations defining classical and modified mathematical models of arc with a previously assumed constant value of coefficient θ = const. Such an approach facilitates the notation of mathematical formulas and, subsequently, the interpretation of the roles of individual components in the approximation of experimental data. In certain cases related to the analysis of circuits operating within a wide range of current amplitude changes, other modifications are assumed, e.g. by applying a non-linear damping functions depending on column current $\theta(i)$ or conductance $\theta(g)$ [9-12]. Because of the adopted initial assumptions and accuracy of approximation, the Mayr model and its modifications are used to represent processes in circuits with low-current arc, whereas the Cassie model and its modifications are used to represent processes in circuits with high-current arc. The parameters of models presented in Table 1 are as follows: θ_M – time constant of the Mayr model or its modification; θ_C – time constant of the Cassie model or its modification; P_M – power of the Mayr model or its modification; U_C – voltage of the Cassie model or its modification; I_W – point abscissa corresponding to ignition voltage in models M2 and M4 ($I_W > 0$A). Similarly, in models C2, C4, C6 and C8, the value of current $I_W > 0$ A reduces the value of arc ignition voltage to zero. In cases of non-linear Schwartz-type models, coefficients of functions approximating power (p_M, α) and voltage (u_C, β) were introduced.

Table 1. Differential equations determining classical and modified models of electric arc (M1, M2 –Mayr-type models; M3, M4 - Mayr-Schwartz-type models; C1, C2 - Cassie-type models; C3, C4 - Cassie-Schwartz-type models; C5, C6 - Cassie-type models with increased dissipation; C7, C8 - Cassie-Schwartz-type models with increased dissipation)

Designation	Classical model with undetermined or unreduced arc ignition voltage	Designation	Modified model with defined or reduced arc ignition voltage
M1	$\theta_M \frac{dg}{dt} + g = \frac{i^2}{P_M}$	M2	$\theta_M \frac{dg}{dt} + g = \frac{i^2 + I_W^2}{P_M} = \frac{i^2}{P_M} + G_{W0}$
M3	$\theta_M \frac{dg}{dt} + g = \frac{i^2}{p_M g^{\alpha}}$	M4	$\theta_M \frac{dg}{dt} + g = \frac{i^2 + I_W^2}{p_M g^{\alpha}}$
C1	$\theta_C \frac{dg^2}{dt} + g^2 = \frac{i^2}{U_C^2}$	C2	$\theta_C \frac{dg^2}{dt} + g^2 = \frac{(\lvert i\rvert + I_W)^2}{U_C^2} = \left(\frac{\lvert i\rvert}{U_C} + G_{W0}\right)^2$
C3	$\theta_C \frac{dg^2}{dt} + g^2 = \frac{i^2}{u_C^2 g^{2\beta}}$	C4	$\theta_C \frac{dg^2}{dt} + g^2 = \frac{(\lvert i\rvert + I_W)^2}{u_C^2 g^{2\beta}}$
C5	$\theta_C \frac{dg^2}{dt} + g^2 = \frac{1}{U_C^2}\left(i - \frac{g}{i} P_{dis}\right)^2$	C6	$\theta_C \frac{dg^2}{dt} + g^2 = \frac{1}{U_C^2}\left(\lvert i\rvert + I_W - \frac{g}{\lvert i\rvert} P_{dis}\right)^2$
C7	$\theta_C \frac{dg^2}{dt} + g^2 = \frac{1}{u_C^2 g^{2\beta}}\left(i - \frac{g}{i} P_{dis}\right)^2$	C8	$\theta_C \frac{dg^2}{dt} + g^2 = \frac{1}{u_C^2 g^{2\beta}}\left(\lvert i\rvert + I_W - \frac{g}{\lvert i\rvert} P_{dis}\right)^2$

Sometimes, in models modifying the Cassie equation, an additional function of dissipation is introduced in order to take into consideration the climbing of current-voltage characteristics within the high-current range. The aforesaid effect is present, among other things, in systems with a small diameter electrode or with high gas pressure in the discharge chamber and is accompanied by intensified heat radiation. The function of power dissipation can have the following form:

$$P_{dis}(i) = a_{i1}|i| + a_{i2}i^2 \qquad (1)$$

or

$$P_{dis}(g) = a_{g1}g + a_{g2}g^2 \qquad (2)$$

Because of the inertia of heat processes in the electric arc column, dependence $P_{dis}(g)$ can be regarded as more rational in comparison with dependence $P_{dis}(i)$ [11]. Table 2 presents formulas of functions describing static current-voltage characteristics of arc, corresponding to the mathematical models presented in Table 1. In most cases, they are described by means of explicit functions. Only as regards the Cassie-Schwartz model (with the additional dissipation of energy C7 and C8), a static characteristic is described using an implicit function.

The modified models were provided with an additional component of current I_W, which, in the case of bipolar excitation, may be responsible for the relatively gentle passage of dynamic and static characteristics through the origin of coordinates (i, u). As a result, the software developer has a certain influence on setting the values of arc ignition voltage and improving the efficiency of the algorithm of the numerical integration of differential equations of the circuit.

As regards model M2, the point of ignition corresponds to coordinates

$$I_W, \qquad \frac{P_M}{2I_W} \qquad (3).$$

When considering model M4, it is possible to obtain expressions enabling the determination of the coordinates of the point of ignition

$$I_W\sqrt{\frac{1+\alpha}{1-\alpha}}, \qquad I_W\sqrt{\frac{1+\alpha}{1-\alpha}}\sqrt[\alpha+1]{\frac{P_M(1-\alpha)}{2I_W^2}} \qquad (4).$$

Table 2. Static characteristics $U(I)$ corresponding to classical and modified models of arc (designations of models as in Table 1)

Designation	Characteristics U(I) of the classical model with undetermined or unreduced arc ignition voltage	Designation	Characteristics U(I) of the modified model with defined or reduced arc ignition voltage
M1	$U = \frac{P_M}{I}$	M2	$U = \frac{P_M I}{I^2 + I_W^2}$
M3	$U = I \cdot \sqrt[\alpha+1]{\frac{p_M}{I^2}}$	M4	$U = I \cdot \sqrt[\alpha+1]{\frac{P_M}{I^2 + I_W^2}}$
C1	$U = U_C \operatorname{sgn} I = U_C \frac{I}{\lvert I \rvert}$	C2	$U = \frac{U_C I}{\lvert I \rvert + I_W}$
C3	$U = I \cdot \sqrt[\beta+1]{\frac{u_C}{\lvert I \rvert}}$	C4	$U = I \cdot \sqrt[\beta+1]{\frac{u_C}{\lvert I \rvert + I_W}}$
C5	$U = \frac{P_{dis}}{I} + U_C \operatorname{sgn} I$	C6	$U = \frac{IU_C + P_{dis} \operatorname{sgn} I}{\lvert I \rvert + I_W}$
C7	$U^\beta (UI - P_{dis}) = u_C I^{\beta+1} \operatorname{sgn} I$	C8	$U^\beta [U(\lvert I \rvert + I_W) - P_{dis} \operatorname{sgn} I] = u_C I^{\beta+1}$

Modifications of functions approximating static characteristics used to build the Pentegov mathematical model of electric arc

A generalised arc column mathematical model satisfying the power balance equation is a model proposed by Pentegov and subsequently developed along with Sidoretz [4]. In the above-named model, instead of actual arc, such hypothetical arc is under consideration, where the arc column conductance is defined as a function of fictitious (virtual) current $i_q(t)$, changing along with specific time constant q [4]. The Pentegov model is represented by a non-linear two-terminal network which is energy-balanced, thermally inert, of first-order, linear, stationary and electrically inertialess

$$\frac{i}{u} = \frac{i_\theta}{U} = g \quad (5).$$

The correlation between the square of state current i_q and the square of arc accrual current i is described by first-order linear differential equation

$$\theta \frac{di_\theta^2}{dt} + i_\theta^2 = i^2 \quad (6).$$

In a general case, arc column voltage is expressed by the following dependence

$$u = \frac{U(i_\theta)}{i_\theta} i \quad (7).$$

In this mode, functions presented in Table 2 are often applied as static characteristics. Examples of similar approximations are presented in [5, 6, 7]. Significantly wider possibilities of the precise approximation of static characteristics are offered by functions presented in Table 3, showing two variants of functions with undetermined or unreduced arc ignition voltage and with defined or reduced arc ignition voltage. Because of the manner in which the characteristics are approximated within the low-current range, they can be rated among the modifications of the Ayrton equation. The approximations presented in Table 4 may have even wider application possibilities. These approximations take into consideration various shapes of characteristics within the low-current range. Because of the manner in which the characteristics are approximated within the low-current range, they can be rated among the modifications of the Nottingham equation.

In cases of static characteristics CA2, CA4 and CA6 as well as CN2, CN4 and CN6, it is possible to determine analytical expressions concerning the coordinates of the point of ignition using an appropriate programme for analytical transformations (e.g. MATHEMATICA). However, because of the fact that they would require considerable volume of the article, related equations have not been presented in the paper. In practical calculations, the values of current I_M and I_W can be equal ($I_M = I_W > 0$A), which could facilitate identifying the value of ignition voltage.

Dynamic current-voltage characteristics of modified models of arc

The investigation of the correct functioning of the developed macromodels of arc required the performance of the simulation of processes in a circuit with the source of sinusoidal current

Table 3. Static characteristics U(I) with undetermined and defined arc ignition voltage

Designation	Characteristics $U(I)$ of the Pentegov model with undetermined or unreduced arc ignition voltage	Designation	Characteristics $U(I)$ of the Pentegov model with defined or reduced arc ignition voltage
CA1	$U = \frac{P_M}{I} + U_C$	CA2	$U_{col} = \frac{P_M I}{I^2 + I_M^2} + \frac{U_C I}{\lvert I \rvert + I_W}$
CA3	$U = \frac{P_M}{I} + U_C + R_p I$	CA4	$U_{col} = \frac{P_M I}{I^2 + I_M^2} + \frac{U_C I}{\lvert I \rvert + I_W} + R_p I$
CA5	$U = \frac{P_M}{I} + U_C + R_p I + k_r I^2 \operatorname{sgn} I$	CA6	$U_{col} = \frac{P_M I}{I^2 + I_M^2} + \frac{U_C I}{\lvert I \rvert + I_W} + R_p I + k_r I^2 \operatorname{sgn} I$

Table 4. Static characteristics $U(I)$ with undetermined and defined arc ignition voltage

Designation	Characteristics $U(I)$ of the Pentegov model with undetermined or unreduced arc ignition voltage	Designation	Characteristics $U(I)$ of the Pentegov model with defined or reduced arc ignition voltage
CN1	$U = U_0\left(\frac{I_0}{I}\right)^n + U_C$	CN2	$U_{col} = U_0\left(\frac{I_0 I}{I^2 + I_M^2}\right)^n + \frac{U_C I}{\lvert I\rvert + I_W}$
CN3	$U = U_0\left(\frac{I_0}{I}\right)^n + U_C + R_p I$	CN4	$U_{col} = U_0\left(\frac{I_0 I}{I^2 + I_M^2}\right)^n + \frac{U_C I}{\lvert I\rvert + I_W} + R_p I$
CN5	$U = U_0\left(\frac{I_0}{I}\right)^n + U_C + R_p I + k_r I^2 \operatorname{sgn} I$	CN6	$U_{col} = U_0\left(\frac{I_0 I}{I^2 + I_M^2}\right)^n + \frac{U_C I}{\lvert I\rvert + I_W} + R_p I + k_r I^2 \operatorname{sgn} I$

having frequency f = 50 Hz. Low-current arc (models M1-M4) was powered by current having an amplitude of 20 A. In the remaining cases, the amplitude of current amounted to 140 A. The adopted value of the sum of near-electrode voltage drops was U_{AK} = 16 V. Selected test results are presented in Figures 1-4. The aforesaid test results justify the conclusion that the proposed mathematical models make it possible to represent various shapes of dynamic characteristics of electric arc.

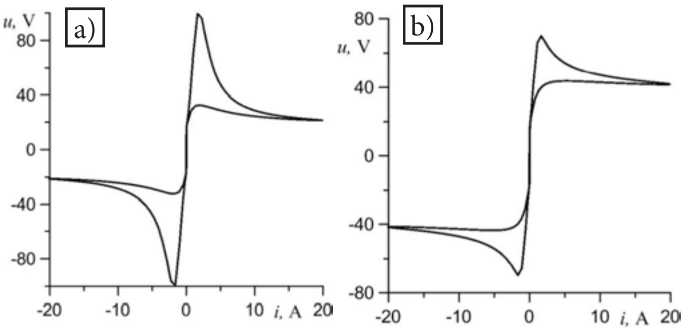

Fig. 1. Dynamic current-voltage characteristics of electric arc: a) described using model M2 (P_M = 100 W, I_W = 0.5 A, θ = 2·10^{-4} s); b) described using model M4 (p_M = 600 V$^{\alpha+1}$/A$^{\alpha-1}$, I_W = 0.5 A, θ = 2·10^{-4} s, α = 0.7)

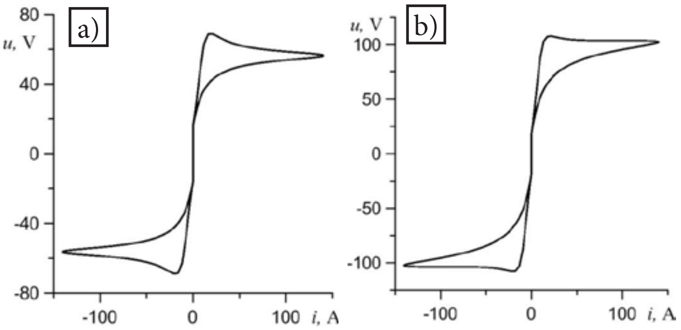

Fig. 2. Dynamic current-voltage characteristics of electric arc: a) described using model C2 (U_C = 40 V, I_w = 1 A, θ = 2·10-4 s); b) described using model C4 (u_C = 80 V$^{\beta+1}$/A$^{\beta}$, I_w = 1 A, θ = 2·10^{-4} s, β = 0.1)

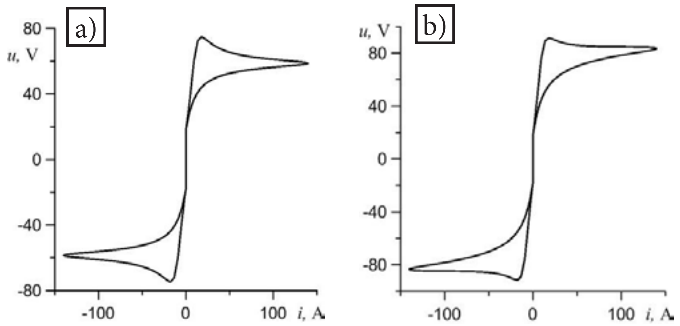

Fig. 3. Dynamic current-voltage characteristics of electric arc: a) described using model C5 (U_C = 40 V, a_{g1} = 0.0001 V^2, a_{g2} = 0.001 V^3/A, θ = 2·10^{-4} s); b) described using model C7 (u_C = 60 V^{b+1}/A^b, b = 0.1, a_{g1} = 0.0001 V^2, a_{g2} = 0.0015 V^3/A, θ = 2·10^{-4} s)

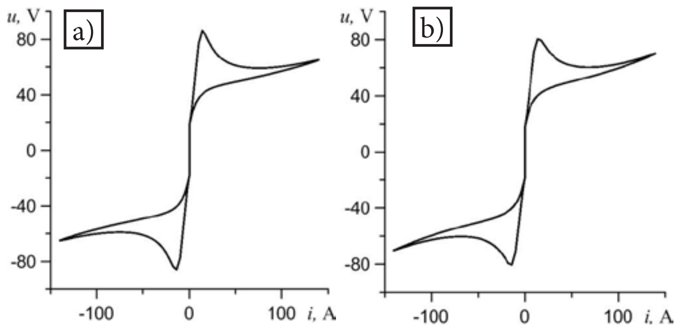

Fig. 4. Dynamic current-voltage characteristics of electric arc described using the Pentegov model: a) with static characteristic CA6 (P_M = 200 W, U_C = 30 V, I_M = 0.2 A, I_W = 0.1 A, R_p = 0.01 Ω, k_r = 0.0007 V/A^2, θ = 2·10^{-4} s) and b) with static characteristic CN6 (U_0 = 100 V, I_0 = 1 A, U_C = 30 V, I_M = 0.2 A, I_W = 0.1 A, R_p = 0.091 Ohm, k_r=0.001 V/A^2, θ = 2·10^{-4} s)

In the cases discussed above, the modifications of the mathematical models of electric arc lead to an increase in the number of parameters of functions approximating static current-voltage characteristics. However, in circuits powered by direct current or bipolar rectangular current of value $\mid i(t)$½= const., it is relatively easy to determine these functions by performing appropriate experimental tests [8, 13, 14]. It is more difficult to determine the value of damping factor *q* of selected models. However, also in this area it is possible to observe significant progress as new, e.g. analytical, integral, linearisation, two-point etc. methods are developed.

Concluding remarks

1. The above-presented set of mathematical models of electric arc makes it possible to relatively easy represent dynamic current-voltage characteristics of arc (low-current and high-current arc as well as arc within a wide range of current changes).
2. The modifications implemented to classical mathematical models enable the obtainment of differential equations easily taking into consideration the preset value of ignition voltage, the numerical solving of which might prove easier because of a decrease in the rate of rise of voltage changes.
3. The above-presented modifications of mathematical models of arc were concerned with static current-voltage characteristics, having a limited effect on difficulties during the experimental determination of parameters.
4. The mathematical models described above can be modified by the variation of the damping factor in the form $\theta(i)$ or $\theta(g)$.

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